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On the equivalence between the impact parameter and transverse momentum space formalisms for the Drell–Yan process

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Abstract. We make extensive use of the Mellin transform technique to effect the Fourier–Bessel transformation between the impact parameter and transverse momentum space formulations of the Drell–Yan process for $q_T^2 \ll q^2$, thereby demonstrating the complete equivalence between the two schemes.

1. Introduction

The Drell–Yan process, in which one observes an l^+l^- pair originating via a massive photon from $q\bar{q}$ annihilation, has been one of the most widely discussed topics within the framework of perturbative QCD in recent years. For large transverse momentum ($q_T^2 \sim q^2$), lowest-order perturbation theory may be used (Kajantie and Raitio 1978, Altarelli *et al* 1978a, b, Fritzsch and Minkowski 1978) to calculate the q_T distribution of the lepton pair. However, in the regime of restricted transverse momentum ($q_T^2 \ll q^2$) resummed perturbation theory must be used to cope with large logarithms of the form $\ln(q_T^2/q^2)$.

The first calculation of the transverse momentum distribution in this regime was performed by Dokshitser *et al* (1980) in the framework of ladder diagrams in momentum space. The result gave the cumulative cross section

$$\Sigma(p_T^2) \equiv \frac{1}{\sigma_0(s)} \int_0^{p_T^2} \frac{d\sigma}{dk_T^2}\, dk_T^2$$

to leading logarithmic (LL) accuracy as the product of the two structure functions evaluated at p_T^2 and a double logarithmic form factor $T^2(p_T^2/q^2)$. Subsequent calculations (Lo and Sullivan 1979, Soper 1980, Ellis and Stirling 1980) showed that the form factor was incorrect and should be replaced to double logarithmic accuracy by $S^2(p_T^2/q^2)$, the square of the Sudakov form factor (SFF). This correction has now been incorporated in the computation, (Dokshitser *et al*), and the final formula given as

$$\Sigma(p_T^2) = \sum_q e_q^2 F_q(x_1, p_T^2) F_{\bar{q}}(x_2, p_T^2) S^2(p_T^2/q^2). \tag{1.1}$$

This structure, including the quark distribution functions, has been verified in lowest-order perturbation theory (Ellis and Stirling 1980, McKitterick 1980, Jones and Wyndham 1980). When full account is taken of the effect of the running coupling constant, S takes on a modified form, given in Ellis and Stirling (1980).

In many ways the formulation in the space of the impact parameter $\boldsymbol{b}$ is more transparent. In particular, the exponentiation of soft-gluon emission occurs directly in $\boldsymbol{b}$ rather than $\boldsymbol{k}_\mathrm{T}$ space. This was the approach taken by Parisi and Petronzio (1979), namely to evaluate the eikonal factor $\chi(b)$ as the Fourier–Bessel transform (FBT) of the lowest-order single-gluon emission cross section, exponentiate and then perform the inverse transform, after integrating k_T^2 up to $p_\mathrm{T}^2 \ll q^2$. To LL accuracy this gives the Sudakov form factor squared, in agreement with (1.1). Including the effect of the distribution functions, Parisi and Petronzio proposed the following expression for the cross section in impact parameter space:

$$\tilde{\sigma}(b, q^2) = \sum_q e_q^2 F_\mathrm{q}(x_1, 1/b^2) F_{\bar{q}}(x_2, 1/b^2) G(b^2, q^2) \tag{1.2}$$

where $G(b^2, q^2)$ is the exponentiated eikonal factor, and $\tilde{\sigma}$ is the FBT of $\sigma_0^{-1}\, \mathrm{d}\sigma/\mathrm{d}q_\mathrm{T}^2$. This formula has subsequently been confirmed in the ladder diagram approach (Jones and Craigie 1980, Collins 1980).

It is clearly important to be able to make the transition between these two formulations. To some extent this was accomplished in Parisi and Petronzio (1979), at least for the double logarithmic form factor, using arguments of a rather heuristic nature. It is the aim of the present paper to set these calculations on a firmer footing, using the machinery of the Mellin transformation, which allows one to control the approximations and in principle to calculate non-leading corrections. The results to leading order agree with those obtained by Parisi and Petronzio, and show the complete equivalence between the two formulations, as given by equations (1.1) and (1.2).

In § 2 we consider the fixed coupling constant case. Here there are two regimes of p_T, separated by $\sqrt{s}\, \mathrm{e}^{-\pi/2\alpha}$: it is only for $p_\mathrm{T} > \sqrt{s}\, \mathrm{e}^{-\pi/2\alpha}$ that equation (1.1) is reproduced. In § 3 we go on to incorporate the running coupling constant appropriate to QCD. Again there are two distinct regimes of p_T, this time separated by the scale parameter Λ. Section 4 concludes the paper with a brief discussion of our results.

2. Fixed coupling constant

2.1. Evaluation of the eikonal factor

In this section we will consider the case of a fixed coupling constant as in QED. Following the procedure used by Parisi and Petronzio (1979), we must first evaluate the eikonal factor denoted by $\chi(b)$, the FBT of the $\mathrm{O}(\alpha)$ contribution to the normalised differential cross section. That is

$$\chi(b) = \frac{1}{\pi} \int_0^s \mathrm{d}^2 \boldsymbol{k}_\mathrm{T} \exp(\mathrm{i}\boldsymbol{k}_\mathrm{T} \cdot \boldsymbol{b}) \nu(k_\mathrm{T}) \tag{2.1}$$

where

$$\nu(k_\mathrm{T}) \equiv (1/\sigma_0(s))\, \mathrm{d}\sigma^{(1)}/\mathrm{d}k_\mathrm{T}^2, \tag{2.2}$$

the superscript (1) referring to the single photon contribution to $\mathrm{d}\sigma/\mathrm{d}k_\mathrm{T}^2$.

Calculation of the lowest-order Feynman diagrams gives

$$\nu(k_\mathrm{T}) = \frac{\alpha}{\pi} \left(\frac{\ln(s/k_\mathrm{T}^2)}{k_\mathrm{T}^2} \right)_+ \tag{2.3}$$

where the subscript '+' is defined by

$$\int_0^s \mathrm{d}k_T^2 \left(\frac{\ln(s/k_T^2)}{k_T^2}\right)_+ f(k_T) = \int_0^s \mathrm{d}k_T^2 \left(\frac{\ln(s/k_T^2)}{k_T^2}\right)(f(k_T) - f(0)). \tag{2.4}$$

Note that (2.1) is not a complete FBT, in the sense that $|\boldsymbol{k}_T|$ only goes up to $\sqrt{s}$ and not infinity. We may rewrite equation (2.1) as

$$\chi(b) = \frac{4\alpha}{\pi}\int_0^1 \mathrm{d}x \frac{\ln x}{x}(1 - J_0(\lambda x)) \tag{2.5}$$

where we have defined $x \equiv k_T/\sqrt{s}$ and $\lambda \equiv b\sqrt{s}$.

A heuristic approach (Parisi and Petronzio 1979) to the evaluation of (2.5) in the relevant region of large λ would be to approximate $J_0(\lambda x)$ as $1 - \theta(x - 1/\lambda)$. The x integral now only goes down to $1/\lambda$, and $\chi(b)$ becomes $-(2\alpha/\pi)\ln^2\lambda$. To verify this result we consider the Mellin transform (MT) of (2.5) with respect to λ:

$$\tilde{f}(j) = \int_0^\infty \mathrm{d}\lambda\, \lambda^j \chi(b, \lambda). \tag{2.6}$$

Consonant with the θ-function argument, we first take the MT of the Bessel function inside the x integral. Integration by parts gives

$$\int_0^\infty \mathrm{d}\lambda\, \lambda^j (1 - J_0(\lambda x)) = -\frac{x^{-(j+1)}}{j+1}\int_0^\infty \mathrm{d}y\, y^{j+1} J_1(y), \tag{2.7}$$

the latter integral being evaluated using (Gradshteyn and Ryzhik 1965)

$$\int_0^\infty \mathrm{d}y\, y^{j+1} J_1(cy) = 2^{j+1} c^{-(j+2)} \frac{\Gamma(\frac{3}{2} + \frac{1}{2}j)}{\Gamma(\frac{1}{2} - \frac{1}{2}j)}, \tag{2.8}$$

analytic for $-3 < j < -\frac{1}{2}$.

We are now left with an x integral of the form

$$\frac{4\alpha}{\pi}\int_0^1 \mathrm{d}x\, x^{-j-2} \ln x = -\frac{4\alpha}{\pi}\frac{1}{(j+1)^2}, \tag{2.9}$$

which is convergent for $j < -1$.

Combining (2.7) and (2.9), we obtain

$$\tilde{f}(j) = \frac{4\alpha}{\pi} 2^{j+1} \frac{\Gamma(\frac{3}{2} + \frac{1}{2}j)}{\Gamma(\frac{1}{2} - \frac{1}{2}j)} \frac{1}{(j+1)^3}. \tag{2.10}$$

In the j plane there is a strip $-3 < j < -1$ for which expression (2.10) is analytic. The limiting form of $f(\lambda)$ as $\lambda \to \infty$ is obtained by closing the contour of the inverse transform $(1/2\pi i)\int \mathrm{d}j\, \lambda^{-j-1}\tilde{f}(j)$ to the right of the strip, picking up the leading triple pole at $j = -1$, which gives $f(\lambda) = -(2\alpha/\pi)\ln^2\lambda$, i.e.

$$\chi(b) = -(\alpha/2\pi)\ln^2(b^2 s). \tag{2.11}$$

Note that since we evaluated $\tilde{f}(j)$ exactly we could have performed the inverse transform as accurately as we like. Though not necessary, one may modify the argument of the logarithm to satisfy formally the boundary condition $\chi(0) = 0$, taking (Parisi and Petronzio 1979) $\chi(b) = -(\alpha/2\pi)\ln^2(1 + b^2 s)$.

2.2. *Inversion of exponentiated form*

Exponentiation of the eikonal factor (2.11) gives the object denoted by $G(b^2, s)$. We now show that the inverse FBT of $G(b^2, s)$, after integration in k_T^2 up to $p_T^2 \ll s$, yields the SFF squared to LL accuracy. The quantity to be evaluated is

$$\Sigma(p_T^2) \equiv \frac{1}{\sigma_0(s)} \int_0^{p_T^2} \frac{d\sigma}{dk_T^2} dk_T^2$$
$$= \frac{1}{4\pi} \int_0^{p_T^2} dk_T^2 \int d^2\boldsymbol{b} \exp(-i\boldsymbol{b} \cdot \boldsymbol{k}_T) \exp[\chi(b)]. \tag{2.12}$$

Following the azimuthal integration, we perform the k_T integration

$$\int_0^{p_T^2} dk_T^2 J_0(k_T b) = 2\frac{J_1(p_T b)}{p_T b}, \tag{2.13}$$

which reduces (2.12) to

$$\Sigma(p_T^2) = \int_0^\infty dx J_1(x) \exp\left[-\frac{\alpha}{2\pi}\ln^2\left(\frac{x^2}{c^2}\right)\right] \tag{2.14}$$

where $x = bp_T$ and $c^2 = p_T^2/s$.

A mean-value argument, based on the support of $J_1(x)$ and the slowly varying nature of $\ln^2(x^2/c^2)$, heuristically leads to an answer $\exp[-(\alpha/2\pi)\ln^2 c^2]$ for (2.14). To verify this result we redefine $y = x/c$ in (2.14) and take the MT with respect to c using (2.8). The remaining y integral is given by

$$\int_0^\infty dy\, y^{-j-2} \exp\left(-\frac{2\alpha}{\pi}\ln^2 y\right) = \frac{\pi}{(2\alpha)^{1/2}} \exp[(j+1)^2\pi/8\alpha]. \tag{2.15}$$

Inverting the MT, we obtain

$$\Sigma(p_T^2) = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} dj\, c^{-(j+1)}\, 2^{j+1} \frac{\Gamma(\frac{3}{2}+\frac{1}{2}j)}{\Gamma(\frac{1}{2}-\frac{1}{2}j)} \frac{\pi}{(2\alpha)^{1/2}} \exp[(j+1)^2\pi/8\alpha] \tag{2.16}$$

where $-3 < \beta < -\frac{1}{2}$.

In the case of a fixed coupling constant there are two regimes to consider: (a) $p_T \to 0$, (b) p_T small but $(\alpha^2/\pi^2)\ln^2(p_T^2/s) < 1$.

(a) To determine the leading term as $p_T \to 0$, we simply pick up the residue from the pole at $j = -3$, giving

$$\Sigma_{p_T^2 \to 0}(p_T^2) = \frac{1}{2}\frac{p_T^2}{s}\frac{\pi}{(2\alpha)^{1/2}} e^{\pi/2\alpha} \tag{2.17}$$

or

$$\frac{1}{\sigma_0(s)}\frac{d\sigma}{dp_T^2}\bigg|_{p_T \sim 0} = \frac{1}{s}\frac{\pi}{(2\alpha)^{1/2}} e^{\pi/2\alpha}, \tag{2.18}$$

which is in slight diagreement with Parisi and Petronzio (1979).

(b) To evaluate (2.16) for p_T small, but

$$p_T/s^{1/2} > e^{-\pi/2\alpha}, \tag{2.19}$$

we use the saddle-point method. Writing $c^{-(j+1)}=e^{-(j+1)\ln c}$, equation (2.16) can be rewritten as

$$\Sigma(p_T^2)=\frac{1}{(8\alpha)^{1/2}}\int_{-\infty}^{+\infty}d\rho\,\exp\left(-i\rho\ln c-\frac{\pi}{8\alpha}\rho^2\right)$$

$$=\frac{1}{(8\alpha)^{1/2}}\int_{-\infty}^{+\infty}d\rho\,\exp\left[-\frac{\pi}{8\alpha}\left(\rho+\frac{4i\alpha}{\pi}\ln c\right)^2\right]\exp\left(-\frac{2\alpha}{\pi}\ln^2 c\right), \tag{2.20}$$

where β was taken as -1, and we have neglected the factor $2^{j+1}[\Gamma(\frac{3}{2}+\frac{1}{2}j)/\Gamma(\frac{1}{2}-\frac{1}{2}j)]$, which does not contribute to the LL result†.

To make the integral into a simple Gaussian we must deform the contour; in doing this we will not meet any poles provided that $(2\alpha/\pi)\,|\ln c|<1$, which is the origin of the constraint (2.19).

Hence (2.20) becomes

$$\Sigma(p_T^2)=\exp\left[-\frac{\alpha}{2\pi}\ln^2\left(\frac{p_T^2}{s}\right)\right] \tag{2.21}$$

which is precisely the square of the SFF, the result being valid for $e^{-\pi/2\alpha}<p_T/\sqrt{s}$.

2.3. *Inclusion of structure functions*

To conclude this section we now include the structure functions. Following Parisi and Petronzio (1979) we take

$$\tilde{F}_{e^-}(x,b^2)\simeq F_{e^-}(x,1/b^2), \tag{2.22}$$

which can be justified in the formulation of Jones and Craigie (1980). For a fixed coupling constant, the structure functions may be represented as

$$F_e(x,q^2)=\sum_n\frac{1}{x^n}A_n\left(\frac{q^2}{m^2}\right)^{-\gamma_n} \tag{2.23}$$

where γ_n is the anomalous dimension of the electron field. Equation (1.2) may now be written as

$$\tilde{\sigma}(b,q^2,s)=\sum_{n,m}\frac{A_nA_m}{x_1^nx_2^m}\sum_q e_q^2(b^2m^2)^{\gamma_n+\gamma_m}e^{\chi(b)}. \tag{2.24}$$

Omitting the sums over n, m and q,

$$\Sigma(p_T^2)=\int_0^{p_T^2}dk_T^2\int d^2\boldsymbol{b}\,\exp(-i\boldsymbol{k}_T\cdot\boldsymbol{b})(b^2m^2)^{\gamma_n+\gamma_m}\exp(\chi(b)) \tag{2.25}$$

Writing $(b^2m^2)^{\gamma_n+\gamma_m}=(p_T^2/m^2)^{-\gamma_n-\gamma_m}(x^2)^{\gamma_n+\gamma_m}$, we can essentially proceed along the same lines as before. After the azimuthal and k_T integrations

$$\Sigma(p_T^2)=\left(\frac{p_T^2}{m^2}\right)^{-\gamma_n-\gamma_m}\int_0^\infty dx\,J_1(x)(x^2)^{\gamma_n+\gamma_m}\exp[-(\alpha/2\pi)\ln^2(x^2/c^2)] \tag{2.26}$$

where $c^2=p_T^2/q^2$.

† In the saddle-point method this factor is evaluated at $j=-1+(4\alpha/\pi)\ln c$, and by (2.19) is therefore of order one.

We again define $y=x/c$, and take the MT with respect to c,

$$\int_0^\infty \mathrm{d}c\, c^{j+1+2(\gamma_n+\gamma_m)} J_1(cy)=2^{j+1+2(\gamma_n+\gamma_m)} y^{-(j+2)-2(\gamma_n+\gamma_m)} \frac{\Gamma(\frac{3}{2}+\frac{1}{2}j+\gamma_n+\gamma_m)}{\Gamma(\frac{1}{2}-\frac{1}{2}j-\gamma_n-\gamma_m)}, \tag{2.27}$$

analytic for $-3-2(\gamma_n+\gamma_m)<j<-\frac{1}{2}-2(\gamma_n+\gamma_m)$.

The y integral is now the same as (2.15). To recover $\Sigma(p_\mathrm{T}^2)$ we take the inverse MT, this being equivalent to (2.20) if we again neglect the factor

$$2^{j+1+2(\gamma_n+\gamma_m)} \frac{\Gamma(\frac{3}{2}+\frac{1}{2}j+\gamma_n+\gamma_m)}{\Gamma(\frac{1}{2}-\frac{1}{2}j-\gamma_n-\gamma_m)}.$$

Restoring the sums over n, m and q gives us equation (1.1) for $\Sigma(p_\mathrm{T}^2)$.

3. Running coupling constant

3.1. Evaluation of eikonal factor

Following the arguments of Parisi and Petronzio (1979), we introduce the running coupling constant by replacing the fixed coupling constant α in equation (2.3) by $(\frac{4}{3})\alpha(k_\mathrm{T}^2)$† where $\alpha(k_\mathrm{T}^2)$ is given (for four flavours) by

$$\alpha(k_\mathrm{T}^2)=12\pi/25\ln(k_\mathrm{T}^2/\Lambda^2). \tag{3.1}$$

The corresponding expression for $\nu(k_\mathrm{T})$ now becomes

$$\nu(k_\mathrm{T})=\frac{16}{25}\left(\frac{1}{k_\mathrm{T}^2}\frac{\ln(q^2/k_\mathrm{T}^2)}{\ln(k_\mathrm{T}^2/\Lambda^2)}\right)_+ \tag{3.2}$$

and hence

$$\chi(b)=\frac{16}{25\pi}\int_{M^2}^{q^2}\frac{\mathrm{d}^2k_\mathrm{T}}{k_\mathrm{T}^2}[\exp(\mathrm{i}\boldsymbol{k}_\mathrm{T}\cdot\boldsymbol{b})-1]\left(\frac{\ln(q^2/k_\mathrm{T}^2)}{\ln(k_\mathrm{T}^2/\Lambda^2)}\right). \tag{3.3}$$

Note that it only makes sense (Parisi and Petronzio 1979) to integrate down to $M^2\gtrsim\Lambda^2$, thus avoiding the singularity, since for $0\leqslant k_\mathrm{T}^2\leqslant M^2$ perturbation theory, and hence expression (3.2), for $\nu(k_\mathrm{T})$, is not valid. Arguing in the same manner as Parisi and Petronzio (1979) (and by using $J_0(x)-1\sim\frac{1}{2}x^2$), we may assume that the contribution from this region is of $\mathrm{O}(b^2M^2)$, and may therefore be neglected in the region of interest, namely

$$1/q^2\ll b^2\ll 1/\Lambda^2. \tag{3.4}$$

Rewriting (3.3) in a form analogous to (2.5), we find

$$\chi(b)=\frac{32}{25}\int_{\varepsilon/\sigma}^1\frac{\mathrm{d}x}{x}\frac{\ln x}{\ln x\sigma}(1-J_0(\lambda x)) \tag{3.5}$$

where $\varepsilon\equiv M/\Lambda\gtrsim 1$, $x^2=k_\mathrm{T}^2/q^2$ and $\sigma^2=q^2/\Lambda^2\to\infty$ is treated as an independent variable from $\lambda^2=b^2q^2$.

† It has been argued by McKitterick (1980) that the relevant momentum scale is k^2, the virtuality of the quark propagating to the quark–photon vertex, rather than k_T^2.

The same heuristic θ-function argument as given in § 2.1 will lead to

$$\chi(b)\sim\frac{16}{25}\ln\left(\frac{q^2}{\Lambda^2}\right)\ln\left(\frac{\ln(1/b^2\Lambda^2)}{\ln(q^2/\Lambda^2)}\right)+\frac{16}{25}\ln(b^2q^2),$$

a result we now verify.

To extract the leading behaviour of $\chi(b)$ we must take the MT with respect to both λ and $\ln\sigma$. However, we must ensure that $\varepsilon/\sigma<1$, implying $\ln\sigma>\ln\varepsilon$.

Repeating the λ integration as in § 2, the double MT now becomes

$$\tilde{f}(j,k)=\frac{-32}{25}\frac{2^{j+1}}{j+1}\frac{\Gamma(\frac{3}{2}+\frac{1}{2}j)}{\Gamma(\frac{1}{2}-\frac{1}{2}j)}\int_{\ln\varepsilon}^{\infty}\mathrm{d}\zeta\,\zeta^k\int_{\varepsilon e^{-\zeta}}^{1}\mathrm{d}x\frac{\ln x}{\ln x+\zeta}x^{-(j+2)},\tag{3.6}$$

where $\zeta=\ln\sigma$, and the analytic strip in the j plane is again $-2<j<-1$.

Inverting the order of the x and ζ integrations leaves a ζ integral of the form

$$\int_{\ln(\varepsilon/x)}^{\infty}\mathrm{d}\zeta\frac{\zeta^k}{\ln x+\zeta}=-\frac{1}{k}\left[\ln\left(\frac{\varepsilon}{x}\right)\right]^k{}_2F_1\left(1,-k;-k+1;\frac{\ln x}{\ln(x/\varepsilon)}\right),\tag{3.7}$$

valid for $k<0$, defining the right-hand boundary of the analytic strip in the k plane. We are left with an x integral of the form

$$\int_0^1\mathrm{d}x\,x^{-(j+2)}\ln x\left[\ln\left(\frac{\varepsilon}{x}\right)\right]^k{}_2F_1\left(1,-k;-k+1;\frac{\ln x}{\ln(x/\varepsilon)}\right).\tag{3.8}$$

At the upper limit, the argument of the hypergeometric function vanishes and the integral is then essentially zero. The pole structure comes from the lower limit, where the argument of the hypergeometric function tends to 1. To pick out the poles of (3.8) we must first write the HGF as a series (Abramowitz and Stegun 1970)

$$\begin{aligned}&{}_2F_1(1,-k;-k+1;z)\\&\quad=\frac{\Gamma(1-k)}{\Gamma(-k)}\sum_{n=0}^{\infty}\frac{(1)_n(-k)_n}{(n!)^2}[\psi(n+1)-\psi(n-k)-\ln(1-z)](1-z)^n\end{aligned}\tag{3.9}$$

and pick out the leading term as $z\to1$, giving

$${}_2F_1\left(1,-k;-k+1;\frac{\ln x}{\ln(x/\varepsilon)}\right)\underset{x\to0}{\approx}k\left[\gamma+\psi(-k)+\ln\left(1+\frac{\ln x}{\ln(\varepsilon/x)}\right)\right].\tag{3.10}$$

The structure of (3.8) is such that we can replace $[\ln(\varepsilon/x)]^k$ by $[\ln(1/x)]^k$ without introducing spurious poles provided we keep to the right of the line $k=-2$. This condition gives us the left-hand boundary of our analytic strip in the k plane. Further, in order not to encounter these spurious $\varepsilon=1$ poles we must close the contour to the right.

Equation (3.8) now becomes

$$\begin{aligned}&-\int_0^1\mathrm{d}x\,x^{-(j+2)}(-\ln x)^{k+1}\left[\gamma+\psi(-k)+\ln\left(\frac{\ln\varepsilon}{\ln(\varepsilon/x)}\right)\right]\\&\quad=-(\gamma+\psi(-k)+\ln\ln\varepsilon)\int_0^{\infty}\mathrm{d}y\,\mathrm{e}^{(j+1)y}y^{k+1}+\int_0^{\infty}\mathrm{d}y\,\mathrm{e}^{(j+1)y}y^{k+1}\ln y\end{aligned}\tag{3.11}$$

where we have put $\varepsilon = 1$ in the second integral, which may be evaluated using (Gradshteyn and Ryzhik 1965)

$$\int_0^\infty x^{\nu-1}\,\mathrm{e}^{-\mu x}\ln x\,\mathrm{d}x = \frac{1}{\mu^\nu}\Gamma(\nu)[\psi(\nu)-\ln\mu], \qquad \operatorname{Re}\mu>0, \operatorname{Re}\nu>0. \tag{3.12}$$

So equation (3.6) now becomes

$$\tilde{f}(j,k) = \frac{32}{25}\frac{2^{j+1}}{j+1}\frac{\Gamma(\frac{3}{2}+\frac{1}{2}j)}{\Gamma(\frac{1}{2}-\frac{1}{2}j)}\frac{\Gamma(k+2)}{[-(j+1)]^{k+2}}$$
$$\times\Big\{-\gamma-\psi(-k)-\ln\ln\varepsilon+\psi(k+2)-\ln[-(j+1)]\Big\}. \tag{3.13}$$

We will first perform the inverse transform in k. By closing the contour to the right and using the recursion relation $\psi(z+1)=\psi(z)+1/z$, we generate a series for $\tilde{f}(j)$:

$$\tilde{f}(j) = -\frac{32}{25}\frac{2^{j+1}}{(j+1)^2}\frac{\Gamma(\frac{3}{2}+\frac{1}{2}j)}{\Gamma(\frac{1}{2}-\frac{1}{2}j)}\sum_{r=1}^{\infty}\frac{\Gamma(r+1)}{[-\zeta(j+1)]^r} \tag{3.14}$$

incorporating the contributions of all the poles for $k \geqslant 0$, which is necessary since the product $\zeta(j+1)$ tends neither to infinity nor zero. However, since $\lambda\to\infty$, we need only pick up the leading poles in the j plane, namely all the poles at $j=-1$, so the expression for $\chi(b)$ becomes

$$\chi(b) = -\frac{32}{25}\zeta\sum_{r=1}^{\infty}\frac{(\ln\lambda)^{r+1}}{r+1}\frac{1}{\zeta^{r+1}}$$
$$=\frac{16}{25}\ln\Big(\frac{q^2}{\Lambda^2}\Big)\ln\Big(\frac{\ln(1/b^2\Lambda^2)}{\ln(q^2/\Lambda^2)}\Big)+\frac{16}{25}\ln(b^2q^2) \tag{3.15}$$

in agreement with Parisi and Petronzio (1979).

3.2. *Inversion of exponentiated form*

To generate the form factor in momentum space, we must now exponentiate (3.15) and evaluate

$$\Sigma(p_{\mathrm{T}}^2) = \frac{1}{4\pi}\int_0^{p_{\mathrm{T}}^2}\mathrm{d}k_{\mathrm{T}}^2\int^{1/\Lambda}\mathrm{d}^2\boldsymbol{b}\,\exp(-\mathrm{i}\boldsymbol{b}\cdot\boldsymbol{k}_{\mathrm{T}})\,\mathrm{e}^{\chi(b)}. \tag{3.16}$$

Note that Σ is always, by definition, integrated from $0\leqslant k_{\mathrm{T}}^2\leqslant p_{\mathrm{T}}^2$, but that $|\boldsymbol{b}|$ now only goes up to $1/\Lambda$ by similar arguments to those in momentum space, p_{T} now being replaced by b^{-1}. The k_{T} and azimuthal integrations in (3.16) are equivalent to those in (2.12), leaving a b integration of the form

$$\Sigma(p_{\mathrm{T}}^2) = \int_0^{p_{\mathrm{T}}/\Lambda}\mathrm{d}x\,J_1(x)\Big(\frac{x^2}{c^2}\Big)^{16/25}\Big(\frac{\ln(\xi^2/x^2)}{\ln(\xi^2/c^2)}\Big)^{16/25\,\ln(\xi^2/c^2)} \tag{3.17}$$

where $c^2 = p_{\mathrm{T}}^2/q^2$, $\xi^2 = p_{\mathrm{T}}^2/\Lambda^2$ and $x = bp_{\mathrm{T}}$ as before.

Following the procedure of § 2.2, we scale out a variable that will ultimately leave an x integration we can evaluate. In (3.17) this variable is ξ, so making the transformation

$y = x/\xi$, (3.17) becomes

$$\Sigma(p_T^2) = \xi \int_0^1 dy\, J_1(\xi y) \left(\frac{\ln y}{\ln \eta}\right)^{-(32/25)\ln\eta} \left(\frac{y}{\eta}\right)^{32/25} \tag{3.18}$$

where $\eta = c/\xi \to 0$ and is treated as an independent variable from ξ.

Taking the MT with respect to ξ using (2.8) leaves a y integration of the form

$$\int_0^1 dy\, (-\ln y)^{-(32/25)\ln\eta} y^{32/25-(k+2)} = [\tfrac{32}{25} - (k+1)]^{-1+(32/25)\ln\eta}\, \Gamma(1 - \tfrac{32}{25}\ln\eta). \tag{3.19}$$

Inverting the MT as in (2.16) gives

$$\Sigma(p_T^2) = \frac{\eta^{-32/25}}{(-\ln\eta)^{-32/25\ln\eta}} \Gamma(1 - \tfrac{32}{25}\ln\eta) \times \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} dk\, \xi^{-(k+1)} 2^{k+1} \frac{\Gamma(\frac{3}{2}+\frac{1}{2}k)}{\Gamma(\frac{1}{2}-\frac{1}{2}k)} [\tfrac{32}{25} - (k+1)]^{-1+32/25\ln\eta} \tag{3.20}$$

valid for $-3 < \beta < -\frac{1}{2}$.

We can evaluate this integral in two momentum regions as in § 2.2.

(a) $p_T \to 0$.

Only the leading pole at $k = -3$ is required; the residue gives

$$\Sigma_{p_T^2\to 0}(p_T^2) = \frac{p_T^2}{2\Lambda^2} \frac{1}{(2A+2)^{1+2A\ln(1/\eta)}} \frac{[\ln(1/\eta)]^{-2A\ln(1/\eta)}}{\eta^{2A}} \Gamma[1 + 2A\ln(1/\eta)], \tag{3.21}$$

where $A = \frac{16}{25}$.

Using Stirling's approximation for the gamma function for large $\ln(1/\eta)$ gives, after differentiation,

$$\frac{1}{\sigma_0}\frac{d\sigma}{dp_T^2}\bigg|_{p_T^2\sim 0} = \frac{1}{\Lambda^2}\left(\frac{\pi}{8}\right)^{1/2} \left(\frac{A^{1/2}}{A+1}\right) \left(\ln\frac{q^2}{\Lambda^2}\right)^{1/2} \left(\frac{\Lambda^2}{q^2}\right)^{A+\ln(1+1/A)} \tag{3.22}$$

in agreement with the saddle-point evaluation of Parisi and Petronzio (1979).

(b) p_T small but $p_T/\Lambda > 1$.

In this case the entire integral must be evaluated since the variable ξ is not small. We again drop the factor $2^{k+1}\,\Gamma(\frac{3}{2}+\frac{1}{2}k)/\Gamma(\frac{1}{2}-\frac{1}{2}k)$, which does not contribute to the LL's, and take $\beta = -1$. Equation (3.20) then becomes

$$\Sigma(p_T^2) = \frac{\eta^{-32/25}}{(-\ln\eta)^{-(32/25)\ln\eta}} \Gamma(1 - \tfrac{32}{25}\ln\eta) \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\rho \frac{e^{i\rho\ln\xi}}{(\frac{32}{25} - i\rho)^{1-(32/24)\ln\eta}} \tag{3.23}$$

where $k + 1 = i\rho$.

Equation (3.23) can be evaluated using the standard integral (Gradshteyn and Ryzhik 1965)

$$\int_{-\infty}^{+\infty} dx \frac{e^{-ipx}}{(\mu - ix)^{\nu}} = \frac{2\pi p^{\nu-1}}{\Gamma(\nu)} e^{-\mu p} \qquad \text{for } p > 0 \tag{3.24}$$

which finally gives

$$\Sigma(p_{\mathrm{T}}^2)=\left(\frac{\ln(q^2/\Lambda^2)}{\ln(p_{\mathrm{T}}^2/\Lambda^2)}\right)^{-(16/25)\ln(q^2/\Lambda^2)}\left(\frac{q^2}{p_{\mathrm{T}}^2}\right)^{16/25} \tag{3.25}$$

in agreement with Ellis and Stirling (1980) and Parisi and Petronzio (1979)†.

3.3. *Inclusion of structure functions*

An advantage of the above method is that we can now derive the momentum space result for equation (1.2) without any additional difficulty.

For the QCD structure functions we use the representation

$$F_{\mathrm{q}}(x,q^2)=\sum_n \frac{1}{x^n}[\ln(q^2/\Lambda^2)]^{-\gamma_n}B_n \tag{3.26}$$

where γ_n is the anomalous dimension of the quark field.

Proceeding along the same lines as before, but now carrying an extra factor $(-\ln b^2\Lambda^2)^{-\gamma_n-\gamma_m}$, (3.18) is modified to

$$\Sigma(p_{\mathrm{T}}^2)=\xi\int_0^1 \mathrm{d}y\, J_1(\xi y)\left(\frac{\ln y}{\ln\eta}\right)^{-(32/25)\ln\eta}\left(\frac{y}{\eta}\right)^{32/25}(-2\ln y)^{-\gamma_n-\gamma_m}. \tag{3.27}$$

Taking the MT with respect to ξ, performing the y integration and inverting the MT gives

$$\Sigma(p_{\mathrm{T}}^2)=\left(\frac{\eta^{-32/25}}{(-\ln\eta)^{-(32/25)\ln\eta}}\right)2^{-\gamma_n-\gamma_m}\Gamma(1-\tfrac{32}{25}\ln\eta-\gamma_n-\gamma_m)$$
$$\times\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}\rho\,\frac{\mathrm{e}^{-\mathrm{i}\rho\ln\xi}}{(\frac{32}{25}-\mathrm{i}\rho)^{1-32\ln\eta/25-\gamma_n-\gamma_m}} \tag{3.28}$$

which, again using (3.24), exactly reproduces the momentum space result (1.1) with S^2 given by (3.25).

4. Conclusions

The infrared resummation necessitated by a restriction of the transverse momentum in the Drell–Yan process is most easily carried out in impact parameter space. However, the results must ultimately be transformed into transverse momentum space in order to make contact with experiment. In this paper we have shown how the required transformations can be systematically evaluated by the use of Mellin transforms, leading to results in complete agreement with those of the more difficult calculations performed directly in p_{T} space.

The Mellin transform has been employed in a somewhat unusual way, in that the contour integral appearing in the inverse Mellin transform has in several places been evaluated exactly rather than in terms of the leading pole. To leading order the net result of some quite involved manipulations is to confirm the remarkably simple

† The direct momentum-space calculation of Ellis and Stirling (1980) leads to an additional, non-leading factor $[\ln(q^2/\Lambda^2)/\ln(p_{\mathrm{T}}^2/\Lambda^2)]^{3/2\times16/25}$. To generate such a term we would require $\nu(k_{\mathrm{T}})$ to next to LL accuracy.

correspondence (Parisi and Petronzio 1979)

$$\Sigma(p_T^2) = \tilde{\sigma}(b^2)|_{b^2=1/p_T^2}. \tag{4.1}$$

Within the Mellin transform framework, the Fourier–Bessel transformations can be evaluated to greater accuracy by picking up the contributions from lower-lying poles and incorporating the effects of the gamma functions. However, it should be borne in mind that such corrections are only meaningful when carried out in parallel with an evaluation of the input $\nu(k_T)$ to a corresponding level of accuracy.

With the appropriate substitutions our results are clearly applicable to the crossed processes $\gamma_v + A \rightarrow B + X$ and $e^+e^- \rightarrow A + B + X$. The same techniques can also be applied to the one-dimensional Fourier transforms arising in the case of the p_{out} distribution in hadron production at large p_T (Jones and Craigie 1980).

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Note added in proof. A general proof of equation (4.1) is in fact contained in a paper by Ralston and Soper (1980). We are grateful to Professor Soper for bringing this reference to our attention. Further discussions of the transformation between impact-parameter and transverse-momentum space are given in recent preprints by Collins and Soper (1980) and Rakow and Webber (1981).

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